Problem 1 (2,5 points) Consider the matrices

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad M' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Suppose that M and M' are the matrices associated with the linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^3$ with respect to different bases:

- M is the matrix associated with f with respect to the bases $\mathcal{B}_1 = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$ and $\mathcal{B}_2 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$.
- M' is the matrix associated with f with respect to the bases $\mathcal{B}_1' = \mathcal{B}_1 = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$ and $\mathcal{B}_2' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$.

Moreover, suppose that $\mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2$. Obtain the matrix of change of bases from $\mathcal{B}_2'$ to $\mathcal{B}_2$.

We are asked to obtain the matrix

$$\underset{B_2 \leftarrow B_2'}{P} = \left([\bar{v}_1]_{B_2} \ [\bar{v}_2]_{B_2} \ [\bar{v}_2]_{B_2} \right).$$

This means that we need to write the elements of B_2' in terms of B_2. We try to obtain this information from the matrices M and M'.

By definition of associated matrices:

$M = \left([f(\bar{e}_1)]_{B_2} \ [f(\bar{e}_2)]_{B_2} \right)$ and

$M' = \left([f(\bar{e}_1)]_{B_2'} \ [f(\bar{e}_2)]_{B_2'} \right).$

So we know how to write $f(\bar{e}_1)$ and $f(\bar{e}_2)$

in terms of $\mathcal{B}_2$ because $[f(\bar{e}_1)]_{\mathcal{B}_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $[f(\bar{e}_2)]_{\mathcal{B}_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ are the columns of M, and how to write $f(\bar{e}_1)$ and $f(\bar{e}_2)$ in terms of $\mathcal{B}_2'$ because $[f(\bar{e}_1)]_{\mathcal{B}_2'} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $[f(\bar{e}_2)]_{\mathcal{B}_2'} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are the columns of M'.

Using this information, we obtain the following equations:

$$f(\bar{e}_1) = \bar{u}_1 + \bar{u}_2 + \bar{u}_3 = \bar{v}_1 + \bar{v}_3$$

$$f(\bar{e}_2) = \bar{u}_1 \qquad\qquad = \qquad \bar{v}_2 + \bar{v}_3 .$$

Together with the equation $\bar{v}_2 = \bar{u}_1 + \bar{u}_2$, now we can find the expressions of $\bar{v}_1, \bar{v}_2, \bar{v}_3$ in terms of $\bar{u}_1, \bar{u}_2, \bar{u}_3$ ($\mathcal{B}_2'$ in terms of $\mathcal{B}_2$).

$$\left.\begin{aligned} \bar{v}_1 \qquad + \bar{v}_3 &= \bar{u}_1 + \bar{u}_2 + \bar{u}_3 \\ \bar{v}_2 \qquad &= \bar{u}_1 + \bar{u}_2 \\ \bar{v}_2 + \bar{v}_3 &= \bar{u}_1 \end{aligned}\right\} \Rightarrow \begin{aligned} \bar{v}_1 &= \bar{u}_1 + 2\bar{u}_2 + \bar{u}_3 \\ \bar{v}_2 &= \bar{u}_1 + \bar{u}_2 \\ \bar{v}_3 &= \quad -\bar{u}_2 \end{aligned}$$

Therefore, $\boxed{\underset{B_2 \leftarrow B'_2}{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}}$

Problem 2 **(2,5 points)** Consider the linear mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that:

$$\ker(f) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x - 2y = 0 \right\}, \ f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \ g \circ f \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Find $g \circ f \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. ¿Is f injective? ¿Is f surjective?

One of many ways to solve this problem is to write

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix} = g\circ f \begin{pmatrix} 5 \\ 2 \end{pmatrix} = g\circ f \left[5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$= 5\, g\circ f \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\, g\circ f \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

because $g\circ f$ is linear. If we are able to find $g\circ f \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then we can solve for $g\circ f \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We now study the information given about f.

A basis for $\ker(f)$ is $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

So $f\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Also $f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (given).

Then

$$\left.\begin{aligned} 2f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + f\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + f\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{aligned}\right\} \Rightarrow \begin{aligned} f\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ f\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 2 \\ -4 \end{pmatrix}. \end{aligned}$$

Notice that $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2} f\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Then

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 5\, g\circ f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\, g\circ f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(\overbrace{-\frac{5}{2} + 2}^{-1/2}\right) g\circ f\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and $\boxed{g\circ f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}}$

Notice that $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^2$ so we know $M_f^{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}$.

$M_f^{\mathcal{B},\mathcal{B}}$ has only one pivot column and one pivot row. So:

$\boxed{f \text{ is neither injective nor surjective}}$

Problem 3 (2,5 points) Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & b & 0 \\ a-1 & 0 & -1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $a, b \in \mathbb{R}$. Determine the values a and b such that $\mathbf{x}$ is an eigenvector of A and A **is not diagonalizable**.

We have two important data:

(1.) $\bar{x}$ is an eigenvector of A.

(2.) A is not diagonalizable.

From (1.) we know

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & b & 0 \\ a-1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ a-1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \leftarrow \lambda = 3 \text{ (row 1)} \\ \\ \leftarrow a-1=0 \text{ (row 3)} \end{matrix}$$

where λ is the eigenvalue associated with $\bar{x}$. This means that $\lambda = 3$ and $a = 1$.

Now we know that $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & b & 0 \\ 0 & 0 & -1 \end{pmatrix}$

and that the eigenvalues are $3, b, -1$ (because A is triangular).

Since A is <u>not diagonalizable</u>, we know that the eigenvalues are not distinct (otherwise A is diagonalizable). So either $b = 3$ or $b = -1$. We study both cases:

<u>Case I $b = 3$:</u>

Since A is not diagonalizable,

$\dim E_3 < 2 \Leftrightarrow \dim E_3 = 1 \Leftrightarrow \operatorname{rank}(A - 3I) = 2.$

(rank$(A-3I)$ $\Leftrightarrow$ only one non-pivot column).

Is this true for $b=3$?

$$\underset{(b=3)}{A-3I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and rank$(A-3I)=2$.

So A is not diagonalizable when $b=3$ and $a=1$.

Case II $b=-1$:

A is not diaganalizable $\Leftrightarrow$ $\dim E_{-1} < 2$

$\Leftrightarrow$ rank$(A+I)=2$.

Is this true for $b=-1$?

$$\underset{(b=-1)}{A+I} = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So rank $(A+I)=1$. This means that A is diagonalizable when $b=-1$ which is a contradiction. Then $b=-1$ is discarded.

In conclusion: $\boxed{a=1 \text{ and } b=3}$

Problem 4 **(2,5 points)** In $\mathbb{R}^4$, consider the following subspaces:

$$U = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 : \begin{array}{c} x-y-z+w=0 \\ x+y-z-w=0 \end{array} \right\}, \quad V_\alpha = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}, \quad \alpha \in \mathbb{R}.$$

Determine the value of the parameter α such that $V_\alpha = U^\perp$. For this value of α, find the orthogonal projection onto $U^\perp$, of the vector $\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ \alpha \\ 4 \end{pmatrix}$.

Remark: Use the usual scalar product in $\mathbb{R}^4$.

We can apprach this problem better if we knew a basis for U.

$$U = \ker \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

In order for V_α to be equal to $U^\perp$, each element of a basis of V_α must be orthogonal to each element of U.

Notice that $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \perp U$. Now, we compute

$$\begin{pmatrix} \alpha & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \alpha - 1$$

$$\begin{pmatrix} \alpha & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 0$$

Therefor $V_\alpha = U^\perp$ if and only if

$$\boxed{\alpha = 1}$$

Now, we compute the orthogonal projection of $\bar{x}$ onto $U^\perp$. We need an orthogonal basis for $U^\perp$. The basis we already have is not orthogonal, so we use Gram-Schmidt to find one:

$$\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\bar{e}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} - \left(\frac{(1 \quad 1 \quad -1 \quad -1)\,\bar{e}_1}{\|\bar{e}_1\|^2} \right) \bar{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

So $\{\bar{e}_1, \bar{e}_2\}$ is an orthogonal basis for $U^\perp$.

The orthogonal projection of $\bar{x}$ onto $U^\perp$ is

$$\hat{x} = \frac{\langle \bar{x}, \bar{e}_1 \rangle}{\|\bar{e}_1\|^2} \bar{e}_1 + \frac{\langle \bar{x}, \bar{e}_2 \rangle}{\|\bar{e}_2\|^2} \bar{e}_2$$

$$= \frac{2}{2} \bar{e}_1 - \frac{4}{2} \bar{e}_2 = \boxed{\begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \end{pmatrix}}$$

Notice that $\hat{x} \perp U$, so this answer is correct.